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Continuous families of coalgebras

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Abstract

Given a small pretopos P, we consider the category $\mathfrak{Mod}(\mathsf{P})$ of models of P indexed over topological spaces. By considering indexed categories of coalgebras, we show that for any indexed functor $F:\mathfrak{Mod}(\mathsf{P})\to\mathfrak{Mod}(\mathsf{Q})$, where Q is another small pretopos, the functor F^1 : $\mathsf{Mod}(\mathsf{P})\to\mathsf{Mod}(\mathsf{Q})$ preserves filtered colimits. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

A pretopos is a category P that is left exact, has strict initial object, stable disjoint finite coproducts and stable quotients of equivalence relations (see [7]). A functor between pretoposes that preserves the structure is called elementary. In [8] Makkai and Reyes explore the relation between pretoposes and first order coherent theories. We can regard a small pretopos P as a first-order coherent theory. The category of models, $\mathsf{Mod}(\mathsf{P})$, for the coherent theory P is the full subcategory of $\mathsf{Set}^{\mathsf{P}}$ whose objects are elementary functors. The category $\mathsf{Mod}(\mathsf{P})$ has filtered colimits and they are preserved by the inclusion $\mathsf{Mod}(\mathsf{P})\to\mathsf{Set}^{\mathsf{P}}$. In general, we cannot guarantee the existence of other kinds of colimits nor can we guarantee the existence of any kind of limit in $\mathsf{Mod}(\mathsf{P})$. However, as a consequence of Łos theorem (see [7]) we have that the ultraproduct in $\mathsf{Set}^{\mathsf{P}}$ of a family of elementary functors is again an elementary functor. That is, $\mathsf{Mod}(\mathsf{P})$ has ultraproducts and they are pointwise.

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Considering filtered colimits, it seems reasonable to ask what extra structure on $\mathsf{Mod}(\mathsf{P})$ and Set would guarantee that a functor $F:\mathsf{Mod}(\mathsf{P})\to\mathsf{Set}$ preserving the extra structure, preserves filtered colimits.

$\mathsf{Mod}(\mathsf{P})$ can be the given structure of an indexed category over the category Top of topological spaces and continuous maps, in the sense of Paré and Schumacher [9], i.e., given a pretopos P define the Top-indexed category $\mathfrak{Mod}(\mathsf{P})$ as follows: For a topological space X, the category $\mathfrak{Mod}(\mathsf{P})^X$ is the full subcategory of $Sh(X)^{\mathsf{P}}$ whose objects are elementary functors, where $Sh(X)$ is the category of sheaves over X. Given another topological space Y and a continuous function $f:Y\to X$, define $f^*:\mathfrak{Mod}(\mathsf{P})^X\to\mathfrak{Mod}(\mathsf{P})^Y$ by composition with the usual $f^*:Sh(X)\to Sh(Y)$. This definition works because $f^*:Sh(X)\to Sh(Y)$ is an elementary functor. The category Set can also be indexed. Denote by $\mathfrak{Set}$ the Top-indexed category such that $\mathfrak{Set}^X=Sh(X)$ and $f^*:\mathfrak{Set}^X\to\mathfrak{Set}^Y$ is the usual $f^*:Sh(X)\to Sh(Y)$. Then $\mathfrak{Set}$ is the category of sets suitably topologized (see [4]). Notice that $\mathfrak{Mod}(\mathsf{P})^1=\mathsf{Mod}(\mathsf{P})$ and $\mathfrak{Set}^1=\mathsf{Set}$. We show that for any Top-indexed functor $F:\mathfrak{Mod}(\mathsf{P})\to\mathfrak{Set}$ we have that the functor F^1:$\mathsf{Mod}(\mathsf{P})\to\mathsf{Set}$ preserves filtered colimits. To do this we generalize a result of Lever.

Lever in [5] showed that for any Top-indexed functor $F:\mathfrak{Set}\to\mathfrak{Set}$, the functor $F^1:\mathsf{Set}\to\mathsf{Set}$ preserves filtered colimits. Furthermore, the assignment $F\mapsto F^1$ is an equivalence $(\,)^1:\mathsf{Top}\text{-}ind(\mathfrak{Set},\mathfrak{Set})\to\mathsf{Filt}(\mathsf{Set},\ \mathsf{Set})$ where Top-*ind* is the category of categories indexed over Top and Filt is the category of categories with filtered colimits and filtered colimit preserving functors. We generalize the result in the following way: Given a category A with filtered colimits and products we construct a Top-indexed category $\mathfrak{A}$. For a topological space X, the category $\mathfrak{A}^X$ is the category of coalgebras for a comonad defined on $\mathsf{A}^{|X|}$ where $|X|$ is the underlying set of the space X. We will have that $\mathfrak{A}^1=\mathsf{A}$ (see the definition below). For $\mathsf{A}=\mathsf{Set}$ we obtain $\mathfrak{A}=\mathfrak{Set}$. We show that given categories A and B with filtered colimits and products in which the filtered colimits satisfy an extra condition with absolute equalizers and products, if $\mathfrak{A},\mathfrak{B}$ are their corresponding Top-indexed categories and $F:\mathfrak{A}\to\mathfrak{B}$ is a Top-indexed functor then $F^1:\mathsf{A}\to\mathsf{B}$ preserves filtered colimits and the functor $(\,)^1:\mathsf{Top}\text{-}ind(\mathfrak{A},\mathfrak{B})\to\mathsf{Filt}(\mathsf{A},\mathsf{B})$ is an equivalence of categories. We follow the same strategy for the proof as the one in [5].

When we apply the above construction to a presheaf category $\mathsf{A}=\mathsf{Set}^{\mathsf{P}}$ we denote the result by $\mathfrak{Set}^{\mathsf{P}}$. With P a pretopos we will have $\mathfrak{Mod}(\mathsf{P})^X$ a full subcategory of $(\mathfrak{Set}^{\mathsf{P}})^X$ with $f^*:\mathfrak{Mod}(\mathsf{P})^X\to\mathfrak{Mod}(\mathsf{P})^Y$ the restriction of $f^*:(\mathfrak{Set}^{\mathsf{P}})^X\to(\mathfrak{Set}^{\mathsf{P}})^Y$. This observation will allow us to prove that for any Top-indexed functor $F:\mathfrak{Mod}(\mathsf{P})\to\mathfrak{Set}$ we have that $F^1:\mathsf{Mod}(\mathsf{P})\to\mathsf{Set}$ preserves filtered colimits.

In the general case we take full subcategories A_0 of A and B_0 of B where A and B satisfy the conditions mentioned above. Under some closure conditions on these subcategories we obtain subTop-indexed categories $\mathfrak{A}_0,\mathfrak{B}_0$ of $\mathfrak{A}$ and $\mathfrak{B}$. In this case, for any Top-indexed functor $F:\mathfrak{A}_0\to\mathfrak{B}_0$ we have that $F^1:\mathsf{A}_0\to\mathsf{B}_0$ preserves filtered colimits.

It is worthwhile to note that ultraproducts play a central role throughout the paper. Given a filter $(I,\mathscr{F})$, that is to say, a set I and a filter $\mathscr{F}$ on I, and a category A

with filtered colimits and products we define the reduced product functor $\prod_{\mathscr{F}} : \mathsf{A}^I \to \mathsf{A}$ such that for any $\langle a_i\rangle_I : \langle A_i\rangle_I \to \langle A_i'\rangle_I$ in A^I we have $\prod_{\mathscr{F}} \langle A_i\rangle_I = \varinjlim_{J\in\mathscr{F}} \prod_{j\in J} A_j$, and $\prod_{\mathscr{F}} \langle a_i\rangle_I = \varinjlim_{J\in\mathscr{F}} \prod_{j\in J} a_j$. In particular, when we have an ultrafilter $(I,\mathscr{U})$ and a Top-indexed functor $F:\mathfrak{A}\to\mathfrak{B}$ we obtain a natural isomorphism $\gamma_{F\mathscr{U}} : F^1 \circ \prod_{\mathscr{U}} \to \prod_{\mathscr{U}} \circ F^I$. From these natural isomorphisms we can recover the indexed functor F uniquely. The same can be said for a Top-indexed functor $F:\mathfrak{A}_0\to\mathfrak{B}_0$.

2. Continuous families of coalgebras

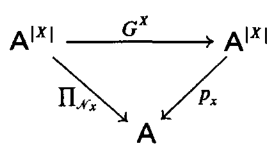
2.1. Notation

All through the paper we assume that A and B are categories with filtered colimits and products, X, Y are topological spaces and $f:Y\to X$ is a continuous function. Given a point $x\in X$ denote by $\mathcal{O}_x = \{U\subset X \mid U \text{ is an open neighborhood of } x\}$. Denote by $\mathcal{N}_x = \{J\subset X \mid J \text{ is a neighborhood of } x\}$. $\mathcal{N}_x$ is clearly a filter on $|X|$. Notice that $\prod_{\mathcal{N}_x}(\langle A_x\rangle) \simeq \varinjlim_{U\in\mathcal{O}_x} \prod_{y\in U} A_y$ for any $\langle A_x\rangle$ in A^I. This means that we can restrict to open sets when using the universal property of the colimit to define an arrow out of $\prod_{\mathcal{N}_x}(\langle A_x\rangle)$. Given a filter $(I,\mathscr{F})$ and an object $\langle A_i\rangle_I$ in A^I, we denote its image under the reduced product functor $\prod_{\mathscr{F}}$ by $\prod A_i/\mathscr{F}$.

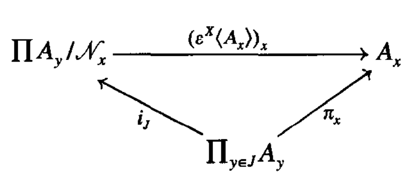
2.2. Continuous families of coalgebras

The definition of the cotriples we need is a direct generalization of the cotriples whose coalgebras are categories of sheaves over topological spaces.

Definition 2.1. Define the cotriple $\mathsf{G}^X = (G^X, \varepsilon^X, \delta^X)$ over $\mathsf{A}^{|X|}$ as follows: The functor $G^X : \mathsf{A}^{|X|} \to \mathsf{A}^{|X|}$ is the unique functor such that for every $x\in X$ the triangle



commutes, where p_x is the xth projection. Define $\varepsilon^X : G^X \to 1$ such that the xth component $(\varepsilon^X\langle A_x\rangle)_x$ of $\varepsilon^X\langle A_x\rangle$ is the unique map that makes the diagram



commute for every $J \in \mathcal{N}_x$. Now, we define $\delta^X : G^X \to G^X G^X$. Let $x \in X$ and $U \in \mathcal{O}_x$. Induce the unique map $\zeta_U : \prod_{u\in U} A_u \to \prod_{u \in U} (\prod A_r/\mathcal{N}_u)$ that makes the diagram

$$\begin{array}{ccc} \prod_{u\in U} A_u & \xrightarrow{\zeta_U} & \prod_{u\in U}(\prod A_r/\mathcal{N}_u) \\ {\scriptstyle i_U}\downarrow & \swarrow{\scriptstyle \pi_u} & \\ \prod A_r/\mathcal{N}_u & & \end{array}$$

commute for every $u \in U$. Notice that we need U open so that it is also a neighborhood of u. Define the xth component

$$(\delta^X \langle A_x \rangle)_x : \prod A_u/\mathcal{N}_x \to \prod\Big(\prod A_r/\mathcal{N}_u\Big) \Big/ \mathcal{N}_x$$

of $\delta^X\langle A_x\rangle$ as the arrow determined by $\varinjlim_{U \in \mathcal{O}_x} \zeta_U$.

It is not hard to see that $(G^X, \varepsilon^X, \delta^X)$ is indeed a cotriple. As a matter of fact, this cotriple is induced by the adjunction $\mathsf{A}^{|X|} \underset{R}{\overset{S}{\rightleftarrows}} \mathsf{A}^{\mathcal{O}(X)^{\mathrm{op}}}$, where $\mathcal{O}(X)$ is the category of opens of X with inclusions as arrows, S is the stalks functor: for $\sigma : F \to F'$ in $\mathsf{A}^{\mathcal{O}(X)^{\mathrm{op}}}$, we have $S(F) = \langle \varinjlim_{U \ni x} FU \rangle_x$ and $S(\sigma) = \langle \varinjlim_{U\ni x} \sigma_U\rangle_x$; and R is such that for any $\langle f_x \rangle_x : \langle A_x\rangle_x \to \langle A_x \rangle_x$ in $\mathsf{A}^{|X|}$ we have $R\langle A_x\rangle_x(U) = \prod_{x\in U} A_x$ and $R\langle f_x\rangle_x(U) = \prod_{x \in U} f_x$ (see [4]).

Definition 2.2. The Top-indexed category $\mathfrak{A}$ is defined as follows: $\mathfrak{A}^X$ is the category $(\mathsf{A}^{|X|})_{\mathsf{G}^X}$ of G^X coalgebras. Let $\langle \tau_x \rangle_x : \langle A_x \rangle_x \to \langle \prod A_u/\mathcal{N}_x \rangle_x$ be a coalgebra in $\mathfrak{A}^X$. Given $y \in Y$ and $J \in \mathcal{N}_{fy}$ we have that $f^{-1}J \in \mathcal{N}_y$. There is a unique arrow

$$\xi_y : \prod A_w/\mathcal{N}_{fy} \to \prod A_{fv}/\mathcal{N}_{fy}$$

that makes the diagram

$$\begin{array}{ccc} \prod A_w/\mathcal{N}_{fy} & \xrightarrow{\xi_y} & \prod A_{fv}/\mathcal{N}_y \\ {\scriptstyle i_J}\uparrow & & \uparrow{\scriptstyle i_{f^{-1}J}} \\ \prod_{w\in J} A_w & \xrightarrow[\langle \pi_{fv}\rangle]{} & \prod_{v \in f^{-1}J} A_w \end{array}$$

commute for every $J \in \mathcal{N}_{fy}$, where $\pi_{fv} : \prod_{w\in J} A_w \to A_{fv}$ is the projection. Let $f^*(\langle \tau_x \rangle_x) = \langle \xi_y \circ \tau_{fy}\rangle_y : \langle A_{fy}\rangle_y \to \langle \prod A_{fv}/\mathcal{N}_{fy}\rangle_y$. Given an arrow $\langle a_x \rangle : \langle \tau_x \rangle \to \langle \tau'_x \rangle$ in $\mathfrak{A}^X$ define $f^*(\langle a_x\rangle_x) = \langle a_{fy}\rangle_y$.

It is not hard to show that this does define a functor $f^* : \mathfrak{A}^X \to \mathfrak{A}^Y$. Furthermore, $\mathfrak{A}$ is then a strict Top-indexed category, that is to say, all the coherence isomorphisms are identities.

We could have defined the category at X to be the full subcategory $Sh_{\mathsf{A}}(X)$ of $\mathsf{A}^{\mathcal{O}(X)^{\mathrm{op}}}$ whose functors satisfy the usual exactness condition. Under the assumption that A is complete, we obtain a right adjoint $\Psi : \mathfrak{A}^X \to \mathsf{A}^{\mathcal{O}(X)^{\mathrm{op}}}$ to the comparison functor $\Phi : \mathsf{A}^{\mathcal{O}(X)^{\mathrm{op}}} \to \mathfrak{A}^X$ (see [2]). It is shown in [10] that for any presheaf F we have that $\Psi\Phi F$ is a sheaf. Furthermore, in the same paper it is shown that the composition $\Psi\Phi$ is left adjoint to the inclusion $Sh_{\mathsf{A}}(X) \to \mathsf{A}^{\mathcal{O}(X)^{\mathrm{op}}}$ if the stalks functor $S : Sh_{\mathsf{A}}(X) \to \mathsf{A}^{|X|}$ reflects isomorphisms and filtered colimits of monomorphisms are monomorphisms in A. This reflector is used in [4] to define $f^* : Sh_{\mathsf{A}}(X) \to Sh_{\mathsf{A}}(Y)$ for any continuous function $f : Y \to X$. Also from [10], if the stalks functor $S : Sh_{\mathsf{A}}(X) \to \mathsf{A}^{|X|}$ preserves equalizers of S-split pairs then $\Psi : \mathfrak{A}^X \to \mathsf{A}^{\mathcal{O}(X)^{\mathrm{op}}}$ is tripleable, and as a consequence, the categories $Sh_{\mathsf{A}}(X)$ and $\mathfrak{A}^X$ are equivalent.

We however do not require A to have equalizers, and pursue our investigation with the categories of coalgebras as stated above. Observe that the definition of f^* is fairly straightforward in this case. We point out that the condition that the stalks functor $S : Sh_{\mathsf{A}}(X) \to \mathsf{A}^{|X|}$ preserves equalizers of S-split pairs is closely related to the condition given in Section 2.4 below, relating filtered colimits and absolute equalizers that we do require A to satisfy.

When A is the category $\mathbf{Set}$, we have that $\mathfrak{A}$ is equivalent, as an indexed category, to $\mathfrak{Set}$.

2.3. Examples of coalgebras

Let us take a look at several topological spaces and their corresponding coalgebras.

Let $(I, \mathscr{F})$ be a filter. Define the topological space $I_{\mathscr{F}}$ whose set of points is $I \cup \{\infty_{\mathscr{F}}\}$, with $\infty_{\mathscr{F}} \notin I$. The topology given by $U \subset I \cup \{\infty_{\mathscr{F}}\}$ open iff $[\infty_{\mathscr{F}} \in U$ implies $U - \{\infty_{\mathscr{F}}\} \in \mathscr{F}]$. $\mathfrak{A}^{I_{\mathscr{F}}}$ is equivalent to the category whose objects are arrows $\tau : A_{\infty_{\mathscr{F}}} \to \prod A_i/\mathscr{F}$, and whose morphisms $\tau \to \tau'$ are families $\langle f_{\infty_{\mathscr{F}}}, \langle f_i \rangle\rangle : \langle A_{\infty_{\mathscr{F}}}, \langle A_i \rangle\rangle \to \langle A'_{\infty_{\mathscr{F}}}, \langle A'_i \rangle\rangle$ that make the square

$$\begin{array}{ccc} A_{\infty_{\mathscr{F}}} & \xrightarrow{\ \tau\ } & \prod A_i/\mathscr{F} \\ {\scriptstyle f_{\infty_{\mathscr{F}}}}\downarrow & & \downarrow{\scriptstyle \prod f_i/\mathscr{F}} \\ A'_{\infty_{\mathscr{F}}} & \xrightarrow[\ \tau'\ ]{} & \prod A'_i/\mathscr{F} \end{array}$$

commute. Assume now we have another filter $\mathscr{E}$ over the same set I such that $\mathscr{F} \subset \mathscr{E}$. We define the continuous function $h_{\mathscr{E}\mathscr{F}} : I_{\mathscr{E}} \to I_{\mathscr{F}}$ such that $h_{\mathscr{E}\mathscr{F}}(\infty_{\mathscr{E}}) = \infty_{\mathscr{F}}$ and $h_{\mathscr{E}\mathscr{F}}(i) = i$ for all $i \in I$. In the description given above the action of $h^*_{\mathscr{E}\mathscr{F}} : \mathfrak{A}^{I_{\mathscr{F}}} \to \mathfrak{A}^{I_{\mathscr{E}}}$ is as follows. The image of $\tau : A_{\infty_{\mathscr{F}}} \to \prod A_i/\mathscr{F}$ is the composition $A_{\infty_{\mathscr{F}}} \xrightarrow{\tau} \prod A_i/\mathscr{F} \to \prod A_i/\mathscr{E}$, where the second arrow makes the diagram

$$\begin{array}{ccc} \prod A_i/\mathscr{F} & \longrightarrow & \prod A_i/\mathscr{E} \\ {\scriptstyle i_J}\nwarrow & & \nearrow{\scriptstyle i_J} \\ & \prod_{j\in J} A_j & \end{array}$$

commute for every $J\in\mathscr{F}$. Given $J_0\subset I$, denote by $\mathscr{S}(J_0)$ the filter generated by J_0. Notice that $\prod A_i/\mathscr{S}(J)\simeq\prod_{j\in J_0}A_j$.

Let X be a topological space and $x_0\in X$. Let $I=|X|-\{x_0\}$ and define $\mathscr{F}_{x_0}=\{J-\{x_0\}\mid J\in\mathscr{N}_{x_0}\}$. $\mathscr{F}_{x_0}$ is a (possibly degenerate) filter on I. Denote $I_{\mathscr{F}_{x_0}}$ by I_{x_0}. We have a continuous function $h_{x_0}:I_{x_0}\to X$ such that $h_{x_0}(\infty)=x_0$ and $h_{x_0}(u)=u$ for every $u\in I$. For any coalgebra $\langle\tau_x\rangle$ in $\mathfrak{A}^X$ we have that

$$h_{x_0}^*\langle\tau_x\rangle=\left(A_{x_0}\xrightarrow{\tau_{x_0}}\prod A_u/\mathscr{N}_x\to\prod A_u/\mathscr{F}_x\right),$$

where the last arrow is obtained by omitting the factor x in all the products.

Lemma 2.3. *If filtered colimits commute with finite products in* A, *then* τ_{x_0} *can be recovered from the composition* $A_{x_0}\xrightarrow{\tau_{x_0}}\prod A_u/\mathscr{N}_{x_0}\to\prod A_u/\mathscr{F}_x$.

Proof. The condition implies that

$$\prod A_u/\mathscr{N}_{x_0}\xrightarrow{\simeq}A_{x_0}\times\prod A_u/\mathscr{F}_{x_0}.$$

Since the diagram

$$\begin{array}{ccc} A_{x_0} & \xrightarrow{\tau_{x_0}} & \prod A_u/\mathscr{N}_{x_0} \\ & {\scriptstyle 1_{A_{x_0}}}\searrow & \downarrow{\scriptstyle (\varepsilon^X\langle\tau_x\rangle)_{x_0}} \\ & & A_{x_0} \end{array}$$

commutes, the result follows. □

Denote the Sierpinski space by S, i.e., S has two points, 0, 1, and its only nontrivial open set is $\{1\}$. The category $\mathfrak{A}^S$ is isomorphic to $\mathsf{A}^{\to}$. Given $j_0\in J\subset I$ we have a continuous function $h_{j_0J}:S\to I_{\mathscr{S}(J)}$ such that $h_{j_0J}(0)=\infty_{\mathscr{S}(J)}$ and $h_{j_0J}(1)=j_0$. Then $h_{j_0J}^*$ sends $\tau:A_{\infty_{\mathscr{S}(J)}}\to\prod_{j\in J}A_j$ to $A_{\infty_{\mathscr{S}(J)}}\xrightarrow{\tau}\prod_{j\in J}A_j\xrightarrow{\pi_{j_0}}A_{j_0}$.

Let D be a small directed preorder. Denote by $T\mathsf{D}$ the topological space whose points are the objects of D endowed with the Alexandroff topology. That is, $U\subset T\mathsf{D}$ open if and only if U is an up-closed subset of D. Given d in $T\mathsf{D}$ we have that $\mathscr{N}_d$ has a minimum, namely $\{d'\mid d'\geq d\}$. Therefore, $\prod A_d/\mathscr{N}_d\simeq\prod_{d'\geq d}A_{d'}$ for any family $\langle A_d\rangle$. It follows that $\mathfrak{A}^{T\mathsf{D}}$ is isomorphic to A^{D}. Denote by $T'\mathsf{D}$ the topological space obtained from $T\mathsf{D}$ by adding an extra point, ∞. The topology of $T'\mathsf{D}$ consists of those sets U such that $U=\emptyset$ or $[\infty\in U$ and $U-\{\infty\}$ is up-closed in $\mathsf{D}]$. The inclusion $i_{\mathsf{D}}:T\mathsf{D}\to T'\mathsf{D}$ is continuous. For reasons that will become clear in the following section we want the functor $i_{\mathsf{D}}^*:\mathfrak{A}^{T'\mathsf{D}}\to\mathfrak{A}^{T\mathsf{D}}$ to be an equivalence. To prove this we will use Beck's tripleability theorem (see [6]). However, we need an extra condition on A that we introduce in the next section.

If $\mathsf{D}=\mathsf{2}$, the ordered set with two elements, 0 and 1, with $0\le 1$, then $T\mathsf{D}$ is Sierpinski's space S. In the topology of $T'\mathsf{2}$ it is not possible to distinguish the points 1 and ∞. This implies that $i_{\mathsf{2}}^*:\mathfrak{A}^{T'\mathsf{2}}\to\mathfrak{A}^{T\mathsf{2}}$ is an equivalence.

2.4. Absolute equalizers

Let D be a small directed preorder and $H:\mathsf{D}\to\mathsf{A}^{\rightrightarrows}$ be a diagram. Given $d\in\mathsf{D}$ denote Hd by $H_0d \overset{h_0d}{\underset{h_1d}{\rightrightarrows}} H_1d$.

Definition 2.4. We say that filtered colimits respect absolute equalizers in A if for every directed preorder D and any diagram $H:\mathsf{D}\to\mathsf{A}^{\rightrightarrows}$ the following condition is satisfied. If for every $d\in\mathsf{D}$ the pair (h_0d,h_1d) has an absolute equalizer $e_d:E_d\to H_0d$ and the pair

$$\varinjlim_d H_0d \overset{\varinjlim_d h_0d}{\underset{\varinjlim_d h_1d}{\rightrightarrows}} \varinjlim_d H_1d$$

has an absolute equalizer, then the diagram

$$\varinjlim_d E_d \xrightarrow{\varinjlim_d e_d} \varinjlim_d H_0d \overset{\varinjlim_d h_0d}{\underset{\varinjlim_d h_1d}{\rightrightarrows}} \varinjlim_d H_1d$$

is an equalizer.

Even though the condition is rather technical, notice that it is satisfied in any left-exact category with filtered colimits in which filtered colimits commute with finite limits. In particular, any locally finitely presentable category satisfies the condition.

2.5. Coalgebras for the space $T'\mathsf{D}$

We assume that filtered colimits respect absolute equalizers on A. Under this circumstances we show that the category $\mathfrak{A}^{T'\mathsf{D}}$ is equivalent to A^{D}.

Define $L:\mathsf{A}^{\mathsf{D}}\to\mathsf{A}^{|T'\mathsf{D}|}$ such that $L(\{A_d\xrightarrow{\sigma_{dd'}}A_{d'}\}_{d\le d'})=(\varinjlim_d A_d,\langle A_d\rangle_d)$, where the first coordinate corresponds to the point ∞. If

$$\{f_d\}:\{A_d\xrightarrow{\sigma_{dd'}}A_{d'}\}_{d\le d'}\to\{B_d\xrightarrow{\tau_{dd'}}B_{d'}\}_{d\to d'}$$

is an arrow in A^{D}, then define $L(\{f_d\})=(\varinjlim_d f_d,\langle f_d\rangle)$.

Lemma 2.5. *If filtered colimits respect pointwise absolute equalizers and* D *is a small directed poset, then the functor* $L:\mathsf{A}^{\mathsf{D}} \to \mathsf{A}^{|T'\mathsf{D}|}$ *defined above is cotripleable.*

Proof. We use Beck's tripleability theorem (see [6]). First, we need a right adjoint. Define $R:\mathsf{A}^{|T'\mathsf{D}|} \to \mathsf{A}^{\mathsf{D}}$ such that

$$R((A_\infty,\langle A_d\rangle)) = \left\{ A_\infty \times \prod_{d''\geq d} A_{d''} \xrightarrow{A_\infty \times p_{dd'}} A_\infty \times \prod_{d''\geq d'} A_{d''} \right\}_{d\leq d'},$$

where $p_{dd'}$ makes the diagram

$$\begin{array}{ccc} \prod_{d''\geq d} A_{d''} & \xrightarrow{p_{dd'}} & \prod_{d''\geq d'} A^{d''} \\ \downarrow{\scriptstyle \pi_{d''}} & & \downarrow{\scriptstyle \pi_{d''}} \\ A_{d''} & \xrightarrow[1_{A_{d''}}]{} & A_{d''} \end{array}$$

commute for every $d''\geq d$. If $(f_\infty,\langle f_d\rangle):(A_\infty,\langle A_d\rangle)\to(B_\infty,\langle B_d\rangle)$ then

$$R(f_\infty,\langle f_d\rangle) = \left\{ f_\infty \times \prod_{d'\geq d} f_d \right\}.$$

R is right adjoint to L.

L clearly reflects isomorphisms.

Suppose $\{f_d\},\{g_d\}:\{A_d\to A_{d'}\}_{d\leq d'}\to\{B_d\to B_{d'}\}_{d\leq d'}$ is a parallel pair in A^{D} such that $L(\{f_d\}),L(\{g_d\})$ has an absolute equalizer

$$(E_\infty,\langle E_d\rangle) \xrightarrow{(e_\infty,\langle e_d\rangle)} (\varinjlim_d A_d,\langle A_d\rangle) \underset{(\varinjlim_d g_d,\langle g_d\rangle)}{\overset{(\varinjlim_d f_d,\langle f_d\rangle)}{\rightrightarrows}} (\varinjlim_d B_d,\langle B_d\rangle).$$

Projecting we obtain, the following absolute equalizers:

$$E_d \xrightarrow{e_d} A_d \underset{g_d}{\overset{f_d}{\rightrightarrows}} B_d \qquad E_\infty \xrightarrow{e_\infty} \varinjlim_d A_d \underset{\varinjlim_d g_d}{\overset{\varinjlim_d f_d}{\rightrightarrows}} \varinjlim_d B_d$$

where d is any element of D.

Therefore, for every $d \leq d'$ in D we can induce an arrow $E_d \to E_{d'}$ such that

$$\begin{array}{ccc} E_d & \xrightarrow{e_d} & A_d \\ \downarrow & & \downarrow \\ E_{d'} & \xrightarrow[e_{d'}]{} & A_{d'} \end{array}$$

commutes. It is easily seen that we obtain an equalizer diagram

$$\{E_d \to E_{d'}\}_{d \leq d'} \xrightarrow{\{e_d\}} \{A_d \to A_{d'}\}_{d \leq d'} \underset{\{g_d\}}{\overset{\{f_d\}}{\rightrightarrows}} \{B_d \to B_{d'}\}_{d \leq d'}.$$

Since filtered colimits respect pointwise absolute equalizers we obtain that L preserves these equalizers. It is clear that L reflects these equalizers. Therefore, L is cotripleable. $\square$

Denote the comparison morphism by $\Phi_{\mathsf{D}} : \mathsf{A}^{\mathsf{D}} \to \mathfrak{A}^{T'\mathsf{D}}$. It is not hard to see that a pseudo-inverse for the composition $\mathfrak{A}^{T\mathsf{D}} \xrightarrow{\simeq} \mathsf{A}^{\mathsf{D}} \xrightarrow{\Phi_{\mathsf{D}}} \mathfrak{A}^{T'\mathsf{D}}$ is i^* where $i = i_{\mathsf{D}} : T\mathsf{D} \to T'\mathsf{D}$ is the inclusion. Let $\Psi_{\mathsf{D}} = (\mathfrak{A}^{T'\mathsf{D}} \xrightarrow{i^*} \mathfrak{A}^{T\mathsf{D}} \xrightarrow{\simeq} \mathsf{A}^{\mathsf{D}})$.

Corollary 2.6. *If filtered colimits respect pointwise absolute equalizers in* A *and* D *is a small directed poset, then the diagram*

$$\begin{array}{ccc} \mathfrak{A}^{T'\mathsf{D}} & \xrightarrow{\Psi_{\mathsf{D}}} & \mathsf{A}^{\mathsf{D}} \\ & {\scriptstyle \infty^*}\searrow \quad \swarrow {\scriptstyle \varinjlim} & \\ & \mathsf{A} & \end{array}$$

commutes up to isomorphism.

3. Filtered colimits and indexed coalgebras

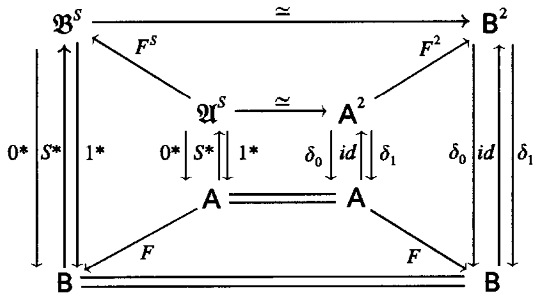
We will assume that in A and in B filtered colimits respect absolute equalizers. We then show that for any Top-indexed functor $F : \mathfrak{A} \to \mathfrak{B}$ the functor $F^1 : \mathsf{A} \to \mathsf{B}$ preserves filtered colimits. To do this it is enough to consider directed colimits (see [1]).

Lemma 3.1. *If* $F : \mathfrak{A} \to \mathfrak{B}$ *is a* Top*-indexed functor, then the diagram*

$$\begin{array}{ccc} \mathfrak{A}^S & \xrightarrow{\simeq} & \mathsf{A}^2 \\ {\scriptstyle F^S}\downarrow & & \downarrow{\scriptstyle F^2} \\ \mathfrak{B}^S & \xrightarrow[\simeq]{} & \mathsf{B}^2 \end{array}$$

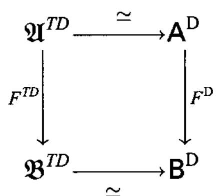
commutes up to isomorphism (where $F^2 = (F^1)^2$*).*

Proof. Consider the diagram $1 \;\substack{\xrightarrow{1} \\ \xleftarrow{S} \\ \xrightarrow[o]{}}\; S$ in Top, where the arrow 1 'picks' $1 \in S$ and the arrow 0 'picks' $0 \in S$. Let $\delta_0, \delta_1 : \mathsf{A}^2 \to \mathsf{A}$ be the domain and codomain functors. Let $id : \mathsf{A} \to \mathsf{A}^2$ be the functor such that $id(A) = 1_A$ and $id(f) = (f, f)$. Consider the diagram



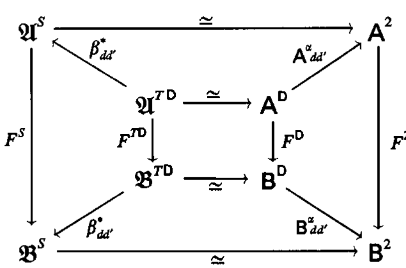
The front and back faces commute sequentially. The right face clearly commutes. Since F is indexed the left face commutes sequentially up to isomorphism. Given an algebra $A_0 \xrightarrow{\langle 1_{A_0}, \alpha \rangle} A_0 \times A_1$ in $\mathfrak{A}^S$, consider the morphism $(1_{A_0}, \alpha) : S^* A_0 \to \langle 1_{A_0}, \alpha \rangle$ in $\mathfrak{A}^S$. Apply F^S and use coherence. □

Lemma 3.2. *If in* A *and in* B *filtered colimits respect absolute equalizers and* $F : \mathfrak{A} \to \mathfrak{B}$ *is a* Top*-indexed functor, then for every small directed preorder* D *the diagram*



commutes up to isomorphism.

Proof. Let $d, d' \in \mathsf{D}$ be such that $d \le d'$. Define the functor $\alpha_{dd'} : \mathbf{2} \to \mathsf{D}$ such that $\alpha_{dd'}(0) = d$ and $\alpha_{dd'}(1) = d'$. Define the continuous function $\beta_{dd'} : S \to T\mathsf{D}$ such that $\beta_{dd'}(0) = d$ and $\beta_{dd'}(1) = d'$. Consider the diagram



Now use Lemma 3.1. □

Corollary 3.3. *If in* A *and in* B *filtered colimits respect absolute equalizers and* $F:\mathfrak{A}\to\mathfrak{B}$ *is a* Top*-indexed functor, then for every small directed preorder* D *the diagram*

$$\begin{array}{ccc} \mathfrak{A}^{T'\mathsf{D}} & \xrightarrow{\Psi_{\mathsf{D}}} & \mathsf{A}^{\mathsf{D}} \\ {\scriptstyle F^{T\mathsf{D}}}\downarrow & & \downarrow{\scriptstyle F^{\mathsf{D}}} \\ \mathfrak{B}^{T'\mathsf{D}} & \xrightarrow[\Psi_{\mathsf{D}}]{} & \mathsf{B}^{\mathsf{D}} \end{array}$$

commutes up to isomorphism.

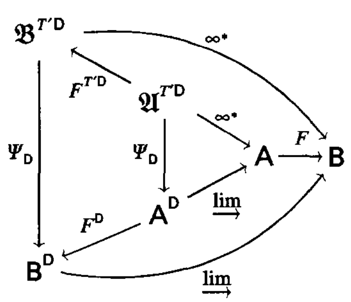
Proof. Since F is a Top-indexed functor the square

$$\begin{array}{ccc} \mathfrak{A}^{T'D} & \xrightarrow{i_{\mathsf{D}}^{*}} & \mathfrak{A}^{\mathsf{D}} \\ {\scriptstyle F^{T'\mathsf{D}}}\downarrow & & \downarrow{\scriptstyle F^{T\mathsf{D}}} \\ \mathfrak{B}^{T'\mathsf{D}} & \xrightarrow[i_{\mathsf{D}}^{*}]{} & \mathfrak{B}^{\mathsf{D}} \end{array}$$

commutes up to isomorphism. Paste this square with the one from Lemma 3.2. □

Theorem 3.4. *If in* A *and in* B *filtered colimits respect absolute equalizers and* $F:\mathfrak{A}\to\mathfrak{B}$ *is a* Top*-indexed functor, then* $F^1:\mathsf{A}\to\mathsf{B}$ *preserves filtered colimits.*

Proof. As we pointed out at the beginning of this section it is enough to consider directed colimits. Let D be a small directed preorder. Consider the diagram

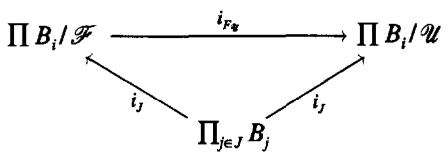


Use Corollaries 3.3 and 2.6. □

This means that we have a functor $(\ )^1 : \mathsf{Top}\text{-}ind(\mathfrak{A}, \mathfrak{B}) \to \mathsf{Filt}(\mathsf{A}, \mathsf{B})$.

4. Reduced products and ultraproducts

Here is another condition we need B to satisfy. Given a filter $(I, \mathscr{F})$, let $\mathscr{C}_{\mathscr{F}} = \{\mathscr{U} \mid \mathscr{U}$ is an ultrafilter on I and $\mathscr{F} \subseteq \mathscr{U}\}$. If $\mathscr{U} \in \mathscr{C}_{\mathscr{F}}$ there is a unique arrow $i_{\mathscr{F}\mathscr{U}} : \prod B_i/\mathscr{F} \to \prod B_i/\mathscr{U}$ making the diagram



commutes for every $J \in \mathscr{F}$.

Definition 4.1. We say that ultraproducts determine reduced products in B if for every filter $(I, \mathscr{F})$ and every family $\langle B_i \rangle$ in B^I the family

$$\langle i_{\mathscr{F}\mathscr{U}} : \textstyle\prod B_i/\mathscr{F} \to \prod B_i/\mathscr{U} \rangle_{\mathscr{U} \in \mathscr{C}_{\mathscr{F}}}$$

is jointly monic.

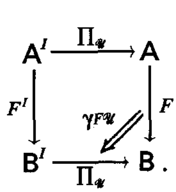
Ultraproducts determine reduced products in Set due to the fact that for every filter $(I, \mathscr{F})$ we have $\mathscr{F} = \bigcap_{\mathscr{U} \in \mathscr{C}_{\mathscr{F}}} \mathscr{U}$.

5. Transition natural transformations

Assume $F : \mathfrak{A} \to \mathfrak{B}$ is a Top-indexed functor. Let $(I, \mathscr{U})$ be an ultrafilter. A coalgebra in $\mathfrak{A}^{I_{\mathscr{U}}}$ is determined by an arrow $\tau : A_\infty \to \prod A_i/\mathscr{U}$ in A. In particular, the arrow $1_{\prod A_i/\mathscr{U}} : \prod A_i/\mathscr{U} \to \prod A_i/\mathscr{U}$ determines a coalgebra in $\mathfrak{A}^{I_{\mathscr{U}}}$. Then

$$F^{I_{\mathscr{U}}}(1_{\prod A_i/\mathscr{U}}) : \infty^*(F^{I_{\mathscr{U}}}(1_{\prod A_i/\mathscr{U}})) \to \prod i^*(F^{I_{\mathscr{U}}}(1_{\prod A_i/\mathscr{U}})).$$

Using the coherence isomorphisms $F\infty^* \to \infty^* F^{I_{\mathscr{U}}}$ and $Fi^* \to i^* F^{I_{\mathscr{U}}}$ we obtain an arrow $\gamma_{F\mathscr{U}}\langle A_i \rangle : F(\prod A_i/\mathscr{U}) \to \prod F A_i/\mathscr{U}$. It turns out that $\gamma_{F\mathscr{U}}$ is a natural transformation as shown



We can actually give an explicit description of $\gamma_{F\mathcal{U}}$. We will need the following lemma

Lemma 5.1. *Given a* Top*-indexed functor* $F:\mathfrak{A}\to\mathfrak{B}$ *and an ultrafilter* $(I,\mathcal{U})$ *we have that for every coalgebra* $\sigma:A_\infty\to\prod A_i/\mathcal{U}$ *the diagram*

$$\begin{array}{ccccc}
F(A_\infty) & & \xrightarrow{\quad F\sigma \quad} & & F(\prod A_i/\mathcal{U}) \\
\downarrow \simeq & & & & \downarrow \gamma F\mathcal{U} \\
\infty^*(F^{I\mathcal{U}}(\sigma)) & \xrightarrow{F^{I\mathcal{U}}(\sigma)} & \prod i^* F^{I\mathcal{U}}(\sigma)/\mathcal{U} & \xrightarrow[\simeq]{} & \prod FA_i/\mathcal{U}
\end{array}$$

commutes.

Proof. Consider the morphism $(\sigma,\langle 1_{A_i}\rangle):\sigma\to 1_{\prod A_i/\mathcal{U}}$ in $\mathfrak{A}^{I\mathcal{U}}$ apply $F^{I\mathcal{U}}$ and use coherence. □

Lemma 5.2. *For any family* $\langle A_i\rangle_I$ *in* A^I *we have that* $\gamma_{F\mathcal{U}}\langle A_i\rangle_I$ *is the composition*

$$F(\textstyle\prod A_i/\mathcal{U}) \xrightarrow{\simeq} \varinjlim_{J\in\mathcal{U}} F\left(\prod_{j\in J} A_j\right) \to \prod FA_i/\mathcal{U},$$

where the isomorphism on the left is due to the fact that F preserves filtered colimits and the arrow on the right is $\varinjlim_{J\in\mathcal{U}}\langle F\pi_j\rangle$ *with* $\pi_j:\prod A_j\to A_j$ *the projection.*

Proof. Let $J\in\mathcal{U}$. Recall from Section 2.3 the definition of the continuous function $h_{jJ}:S\to I_{\mathcal{S}(J)}$ for every $j\in J$. The square

$$\begin{array}{ccc}
\mathfrak{A}^{I_{\mathcal{S}(J)}} & \xrightarrow{h_{jJ}^*} & \mathfrak{A}^S \\
\downarrow F^{\mathcal{S}(J)} & & \downarrow F^{\mathcal{S}} \\
\mathfrak{B}^{I_{\mathcal{S}(J)}} & \xrightarrow[h_{jJ}^*]{} & \mathsf{B}^S
\end{array}$$

commutes up to a coherent isomorphism. Thus, for an algebra $\tau:A_\infty\to\prod_{j\in J}A_j$ in $\mathfrak{A}^{I_{\mathcal{S}(J)}}$ we obtain the commutative diagram

$$\begin{array}{ccccccc}
FA_\infty & \xrightarrow{\simeq} & 0^*F^S(\pi_j\tau) & \xrightarrow{F^S(\pi_j\tau)} & 1^*F^S(\pi_j\tau) & \xrightarrow{\simeq} & FA_j \\
\downarrow\simeq & & & & & & \downarrow\simeq \\
\infty^*F^{I_{\mathcal{S}(J)}}(\tau) & & \xrightarrow[F^{I_{\mathcal{S}(J)}}(\tau)]{} & & \prod_{j\in J} j^*F^{I_{\mathcal{S}(J)}}(\tau) & \xrightarrow[\pi_j]{} & j^*F^{I_{\mathcal{S}(J)}}(\tau)
\end{array}$$

Notice that by Lemma 3.1 the top composition in the above diagram is $F(\pi_j\tau)$. Also, form Section 2.3, recall the definition of the continuous function $h_{\mathscr{U}\mathscr{S}(J)}: I_{\mathscr{U}} \to I_{\mathscr{S}(J)}$. Using the commutative diagram above and the fact that

$$\begin{array}{ccc}
\mathfrak{A}^{I_{S(J)}} & \xrightarrow{h^*_{\mathscr{U}\mathscr{S}}} & \mathfrak{A}^{I_{\mathscr{U}}} \\
{\scriptstyle F^I_{\mathscr{S}(J)}}\Big\downarrow & & \Big\downarrow{\scriptstyle F^{I_{\mathscr{U}}}} \\
\mathfrak{B}^{I_{\mathscr{S}(J)}} & \xrightarrow[h^*_{\mathscr{U}\mathscr{S}(J)}]{} & \mathfrak{B}^{I_{\mathscr{U}}}
\end{array}$$

commutes up to coherent isomorphism, it follows that the diagram

$$\begin{array}{ccccc}
FA_\infty & \xrightarrow{\langle F(\pi_j\tau)\rangle_J} & \prod_{j\in J} FA_j & \xrightarrow{i_J} & \prod FA_i/\mathscr{U} \\
\simeq\Big\downarrow & & & & \Big\downarrow\simeq \\
\infty^* F^{I_{\mathscr{U}}}(i_J\tau) & & \xrightarrow[F^{I_{\mathscr{U}}}(i_J\tau)]{} & & \prod i^* F^{I_{\mathscr{U}}}(i_J\tau)/\mathscr{U}
\end{array}$$

commutes, where $i_J\tau$ is the composition of τ and $i_J : \prod_{j\in J} A_j \to \prod A_i/\mathscr{U}$. Consider the particular case where $\tau = 1_{\prod_{j\in J}A_j}$, and apply $F^{I_{\mathscr{U}}}$ to the morphism

$$\begin{array}{ccc}
\prod_{j\in J} A_j & \xrightarrow{i_J} & \prod A_i/\mathscr{U} \\
{\scriptstyle i_J}\Big\downarrow & & \Big\downarrow{\scriptstyle \prod 1_{A_i/\mathscr{U}}} \\
\prod A_i/\mathscr{U} & \xrightarrow[1_{\prod A_i/\mathscr{U}}]{} & \prod A_i/\mathscr{U}
\end{array}$$

in $\mathfrak{A}^{I_{\mathscr{U}}}$. Then, with the help of Lemma 5.1 and coherence we have that

$$\begin{array}{ccccc}
F(\prod_{j\in J} A_j) & \xrightarrow{\langle F\pi_j\rangle} & \prod_{j\in J} FA_j & \xrightarrow{i_J} & \prod FA_i/\mathscr{U} \\
{\scriptstyle F(i_J)}\Big\downarrow & & & & \Big\downarrow{\scriptstyle 1} \\
F(\prod A_i/\mathscr{U}) & & \xrightarrow[\gamma_{F\mathscr{U}}\langle A_i\rangle]{} & & \prod FA_i/\mathscr{U}
\end{array}$$

commutes. Since we started with an arbitrary $J \in \mathscr{U}$ the result follows. $\square$

Lemma 5.3. *Let $F:\mathfrak{A}\to\mathfrak{B}$ be a* Top*-indexed functor. For any ultrafilter $\mathscr{U}\in\mathscr{C}_{\mathscr{F}}$ and any coalgebra $\sigma:A_{\infty_{\mathscr{F}}}\to\prod A_i/\mathscr{F}$ in $\mathfrak{A}^{I_{\mathscr{F}}}$ the diagram*

$$\begin{array}{ccccccc}
FA_{\infty_{\mathscr{F}}} & \xrightarrow{F\sigma} & F(\prod A_i/\mathscr{F}) & & \xrightarrow{Fi_{\mathscr{U}\mathscr{F}}} & & F(\prod A_i/\mathscr{U}) \\
\simeq\big\downarrow & & & & & & \big\downarrow\gamma_{F\mathscr{U}} \\
\infty_{\mathscr{F}}^{*}F^{I_{\mathscr{F}}}(\sigma) & \xrightarrow[F^{I_{\mathscr{F}}}(\sigma)]{} & \prod i^{*}F^{I_{\mathscr{F}}}(\sigma)/\mathscr{F} & \xrightarrow[i_{\mathscr{F}\mathscr{U}}]{} & \prod i^{*}F^{I_{\mathscr{F}}}(\sigma)/\mathscr{U} & \xrightarrow[\simeq]{} & \prod FA_i/\mathscr{U}
\end{array}$$

commutes.

Proof. Apply Lemma 5.1 to the coalgebra $i_{\mathscr{F}\mathscr{U}}\sigma$. Use the fact that the diagram

$$\begin{array}{ccc}
\mathfrak{A}^{I_{\mathscr{F}}} & \xrightarrow{h^{*}_{\mathscr{U}\mathscr{F}}} & \mathfrak{A}^{I_{\mathscr{U}}} \\
F^{I_{\mathscr{F}}}\big\downarrow & & \big\downarrow F^{I_{\mathscr{U}}} \\
\mathfrak{B}^{I_{\mathscr{F}}} & \xrightarrow[h^{*}_{\mathscr{U}\mathscr{F}}]{} & \mathfrak{B}^{I_{\mathscr{U}}}
\end{array}$$

commutes up to a coherent isomorphism and the coherent isomorphisms arising from the commutative diagrams

$$\begin{array}{ccc}
I_{\mathscr{U}} & \xrightarrow{h_{\mathscr{U}\mathscr{F}}} & I_{\mathscr{F}} \\
{\scriptstyle\infty_{\mathscr{U}}}\nwarrow & & \nearrow{\scriptstyle\infty_{\mathscr{F}}} \\
& 1 &
\end{array}
\qquad\qquad
\begin{array}{ccc}
I_{\mathscr{U}} & \xrightarrow{h_{\mathscr{U}\mathscr{F}}} & I_{\mathscr{F}} \\
{\scriptstyle i}\nwarrow & & \nearrow{\scriptstyle i} \\
& 1 &
\end{array}$$

in Top. □

Recall from Section 2.3 the definitions of $\mathscr{F}_{x_0}$ and $h_{x_0}:I_{x_0}\to X$. With a similar proof we have

Lemma 5.4. *Let X be a topological space and $x_0\in X$. For any coalgebra $\langle\tau_x\rangle$ in $\mathfrak{A}^X$ the diagram*

$$\begin{array}{ccccc}
x_0^{*}F^{X}\langle\tau_x\rangle & \xrightarrow{(F^{X}\langle\tau_x\rangle)_{x_0}} & \prod u^{*}F^{X}\langle\tau_x\rangle/\mathscr{N}_{x_0} & \xrightarrow{\pi} & \prod u^{*}F^{X}\langle\tau_x\rangle/\mathscr{F}_{x_0} \\
\big\downarrow\simeq & & & & \simeq\big\downarrow \\
\infty^{*}F^{I_{x_0}}(\pi\tau_{x_0}) & & \xrightarrow[F^{I_{x_0}}(\pi\tau_{x_0})]{} & & \prod u^{*}F^{I_{x_0}}(\pi\tau_{x_0})/\mathscr{F}_{x_0}
\end{array}$$

commutes, where $(F^X\langle\tau_x\rangle)_{x_0}$ denotes the x_0th component of the coalgebra $F^X\langle\tau_x\rangle$.

6. From filtered colimit preserving functors to indexed functors

We define now a functor $\widehat{(\ )}:\mathsf{Filt}(\mathsf{A},\mathsf{B})\to\mathsf{Top}\text{-}ind(\mathfrak{A},\mathfrak{B})$. Given a colimit preserving functor $H:\mathsf{A}\to\mathsf{B}$, define $\widehat{H}:\mathfrak{A}\to\mathfrak{B}$ as follows. For a topological space X and a coalgebra $\langle\tau_x\rangle:\langle A_x\rangle\to\langle\prod A_u/\mathcal{N}_x\rangle$ in $\mathfrak{A}^X$, define the xth component of $\widehat{H}^X(\langle\tau_x\rangle)$ to be the composition

$$HA_x \xrightarrow{H\tau_x} H\left(\prod A_u/\mathcal{N}_x\right) \xrightarrow{\simeq} \varinjlim_{U\in\mathcal{N}_x} H\left(\prod_{u\in U} A_u\right) \longrightarrow \prod HA_u/\mathcal{N}_x,$$

where the middle isomorphism is due to the fact that H preserves filtered colimits and the last arrow is $\varinjlim_{U\in\mathcal{N}_x}\langle H\pi_u\rangle_{u\in U}$ with $\pi_u:\prod_{u\in U}A_u\to A_u$ the projection. A straightforward diagram chasing shows that $\widehat{H}^X(\langle\tau_x\rangle)$ is a coalgebra in $\mathfrak{B}^X$. Given a morphism $\langle f_x\rangle:\langle\tau_x\rangle\to\langle\tau'_x\rangle$ in $\mathfrak{A}^X$ define $\widehat{H}^X(\langle f_x\rangle)=\langle Hf_x\rangle:\widehat{H}^X(\langle\tau_x\rangle)\to\widehat{H}^X(\langle\tau'_x\rangle)$. Another diagram chasing shows that for every $f:Y\to X$ in Top, the diagram

$$\begin{array}{ccc}
\mathfrak{A}^X & \xrightarrow{f^*} & \mathfrak{A}^Y \\
{\scriptstyle \widehat{H}^X}\downarrow & & \downarrow{\scriptstyle \widehat{H}^Y} \\
\mathfrak{B}^X & \xrightarrow[f^*]{} & \mathfrak{B}^Y
\end{array}$$

commutes on the nose. Thus, $\widehat{H}$ is a strict Top-indexed functor. Given a natural transformation $\theta:H\to H'$ in $\mathsf{Filt}(\mathsf{A},\mathsf{B})$ define $\widehat{\theta}^X\langle\tau_x\rangle=\langle\theta A_x\rangle$. This completes the definition of the functor $\widehat{(\ )}$.

Theorem 6.1. *Let* A *and* B *be categories with products and filtered colimits. Assume that in* A *and in* B *filtered colimits respect absolute equalizers, filtered colimits commute with finite products and that reduced products are determined by ultraproducts in* B. *Then the functor*

$$(\_)^1:\mathsf{Top}\text{-}ind(\mathfrak{A},\mathfrak{B})\to\mathsf{Filt}(\mathsf{A},\mathsf{B})$$

is an equivalence.

Proof. We will show that the functor $\widehat{(\ )}$ defined above is a pseudo-inverse for $(\_)^1$. Clearly, $(\_)^1\circ\widehat{(\ )}=1_{\mathsf{Filt}(\mathsf{A},\mathsf{B})}$. Let $F:\mathfrak{A}\to\mathfrak{B}$ be a Top-indexed functor and X a topological space. We have to define a natural transformation $\varphi^X:F^X\to\widehat{F^1}^X$. Let $\langle\tau_x\rangle$ be a coalgebra in $\mathfrak{A}^X$. Notice that for any $x\in X$ we have $x^*\widehat{F^1}^X\langle\tau_x\rangle=F^1(A_x)=FA_x$. Define the xth component of $\varphi^X\langle\tau_x\rangle$ to be the coherent isomorphism $x^*F^X(\langle\tau_x\rangle)\to FA_x$. We show now that this defines a morphism of coalgebras. Lemmas 5.1 and 5.2 show that this is the case when $X=I_{\mathcal{U}}$ for any ultrafilter $(I,\mathcal{U})$. We consider next the case

$X = I_{\mathscr{F}}$ for a filter $(I, \mathscr{F})$. Let $\sigma : A_\infty \to \prod A_i/\mathscr{F}$ be a coalgebra in $\mathfrak{A}^{I_{\mathscr{F}}}$. Since in B reduced products are determined by ultraproducts it suffices to show that for every ultrafilter $\mathscr{U}$ on I containing $\mathscr{F}$ the diagram

$$\begin{array}{ccccccc}
FA_\infty & \xrightarrow{F\sigma} & F(\prod A_i/\mathscr{F}) & \xrightarrow{\simeq} & \varinjlim_{J\in\mathscr{F}} F(\prod_J A_j) & & \\
\downarrow{\scriptstyle \simeq} & & & & \downarrow & & \\
\infty^* F^{I_{\mathscr{F}}}(\sigma) & \xrightarrow[F^{I_{\mathscr{F}}}(\sigma)]{} & \prod i^* F^{I_{\mathscr{F}}}(\sigma) & \xrightarrow[\simeq]{} & \prod FA_i/\mathscr{F} & \xrightarrow[i_{\mathscr{F}\mathscr{U}}]{} & \prod FA_i/\mathscr{U}
\end{array}$$

commutes, where the right top horizontal arrow is due to the fact that F preserves filtered colimits and the right vertical arrow is induced by the products. The diagram above commutes as a consequence of Lemma 5.3. We now consider the general case. We have to show that for any $x \in X$ the diagram

$$\begin{array}{ccccccc}
x^* F^X\langle\tau_x\rangle & & \xrightarrow{(F^X\langle\tau_x\rangle)_x} & & & & \prod u^* F^X\langle\tau_x\rangle/\mathscr{N}_x \\
\downarrow{\scriptstyle \simeq} & & & & & & \downarrow{\scriptstyle \simeq} \\
FA_x & \xrightarrow[F\tau_x]{} & F(\prod A_u/\mathscr{N}_x) & \xrightarrow[\simeq]{} & \varinjlim_{J\in\mathscr{N}_x} F(\prod_{u\in J} A_u) & \longrightarrow & \prod FA_u/\mathscr{N}_x
\end{array}$$

commutes. It suffices, in face of Lemma 2.3, that the diagram above composed with

$$\pi : \prod FA_u/\mathscr{N}_x \to \prod FA_u/\mathscr{F}_x$$

commutes. This follows from Lemma 5.4.

It is straightforward to show that φ^X is a natural transformation and that φ is indexed over Top. □

7. Subcategories closed under ultraproducts

Suppose now that we have a full subcategory A_0 of A with filtered colimits and such that the inclusion $\mathsf{A}_0 \to \mathsf{A}$ preserves filtered colimits. Define the Top-indexed category $\mathfrak{A}_0$ as follows: For a topological space X, $\mathfrak{A}_0^X$ is the full subcategory of $\mathfrak{A}^X$ whose objects are those coalgebras $\langle\tau_x\rangle : \langle A_x\rangle \to \langle\prod A_u/\mathscr{N}_x\rangle$ such that for every $x \in X$ the object A_x is in A_0. If $f : Y \to X$ is a continuous map $f^* : \mathfrak{A}_0^X \to \mathfrak{A}_0^Y$ is the restriction of $f^* : \mathfrak{A}^X \to \mathfrak{A}^Y$.

Let D be a directed preorder. Recall from Section 2.5 the topological spaces $T\mathsf{D}$ and $T'\mathsf{D}$, the inclusion $i_{\mathsf{D}} : T\mathsf{D} \to T'\mathsf{D}$ and the comparison functor $\Phi_{\mathsf{D}} : \mathsf{A}^{\mathsf{D}} \to \mathfrak{A}^{T'\mathsf{D}}$.

Lemma 7.1. *If filtered colimits respect pointwise absolute coequalizers, then*

$$i^* : \mathfrak{A}_0^{T'\mathsf{D}} \to \mathfrak{A}_0^{T\mathsf{D}}$$

is an equivalence.

Proof. The isomorphism $\mathfrak{A}^{T\mathsf{D}} \to \mathsf{A}^{\mathsf{D}}$ restricts to an isomorphism $\mathfrak{A}_0^{T\mathsf{D}} \to \mathsf{A}_0^{\mathsf{D}}$, and the comparison functor $\Phi_{\mathsf{D}} : \mathsf{A}^{\mathsf{D}} \to \mathfrak{A}^{T'\mathsf{D}}$ also restricts to $\Phi'_{\mathsf{D}} = \Phi_{\mathsf{D}}|_{\mathsf{A}_0} : \mathsf{A}_0^{\mathsf{D}} \to \mathfrak{A}_0^{T'\mathsf{D}}$. □

Assume that in A and in B filtered colimits respect pointwise absolute equalizers. Let A_0 be a full subcategory of A and B_0 be a full subcategory of B closed under filtered colimits. Let $F : \mathfrak{A}_0 \to \mathfrak{B}_0$ be a Top-indexed functor. Notice that all the propositions of Section 3 remain true if we replace $\mathfrak{A}$ and $\mathfrak{B}$ by $\mathfrak{A}_0$ and $\mathfrak{B}_0$. In particular,

Theorem 7.2. *With the above notation, the functor $F^1 : \mathsf{A}_0 \to \mathsf{B}_0$ preserves filtered colimits.*

Definition 7.3. With A and A_0 as above we say that A_0 is closed under A-ultraproducts if for every ultrafilter $(I, \mathscr{U})$ the functor $\prod_{\mathscr{U}} : \mathsf{A}^I \to \mathsf{A}$ restricts to $\prod_{\mathscr{U}} : \mathsf{A}_0^I \to \mathsf{A}_0$.

Notice that, when the subcategories A_0 and B_0 are closed under ultraproducts we still obtain natural transformations $\gamma_{F\mathscr{U}}$ as in Section 5. With the same proofs we have, replacing $\mathfrak{A}$ and $\mathfrak{B}$ with $\mathfrak{A}_0$ and $\mathfrak{B}_0$, Lemmas 5.1, 5.3 and 5.4.

We do not get and explicit description of the transformations $\gamma_{F\mathscr{U}}$ nor are we able to construct a Top-indexed functor as before due to the fact that we are not assuming that A_0 or B_0 have products.

8. Categories of models

All the conditions we have imposed on the category A are satisfied by any presheaf category. In particular, let us consider $\mathsf{Set}^{\mathsf{P}}$ for a small pretopos P. Denote the Top-indexed category of coalgebras for this category by $\mathfrak{Set}^{\mathsf{P}}$. We have the full subcategory $\mathsf{Mod(P)}$ of $\mathsf{Set}^{\mathsf{P}}$ of models. Since $\mathsf{Mod(P)}$ is closed under filtered colimits we can carry out the construction of Section 7, denote the resulting category by $\mathfrak{Mod}^{\mathsf{P}}$. Recall the definition of the Top-indexed category $\mathfrak{Mod}(\mathsf{P})$ form Section 1.

Lemma 8.1. *The* Top*-indexed categories* $\mathfrak{Mod}(\mathsf{P})$ *and* $\mathfrak{Mod}^{\mathsf{P}}$ *are equivalent.*

Proof. Let X be a topological space. Given a coalgebra

$$\langle \tau_x \rangle : \langle M_x \rangle \to \langle \prod M_u / \mathscr{N}_x \rangle$$

in $(\mathfrak{Mod}^{\mathsf{P}})^X$ we obtain a functor $\mathsf{P} \to Sh(X)$ defined by

$$P \mapsto \langle \tau_x P \rangle : \langle M_x P \rangle \to \left\langle \prod M_u P / \mathscr{N}_x \right\rangle .$$

Conversely, given an elementary functor $M : \mathsf{P} \to Sh(X)$ we obtain a coalgebra

$$\langle \sigma_x \rangle : \langle x^* M \rangle \to \left\langle \prod u^* M/\mathcal{N}_x \right\rangle$$

such that $\sigma_x P$ is the xth component of $MP : \langle x^* MP \rangle \to \langle \prod u^* MP/\mathcal{N}_x \rangle$. □

As a corollary to Theorem 7.2 we have

Theorem 8.2. *Given small pretoposes* P *and* Q, *and a* Top-*indexed functor*

$$F : \mathfrak{Mod}(\mathsf{P}) \to \mathfrak{Mod}(\mathsf{Q}),$$

the functor $F^1 : \mathsf{Mod}(\mathsf{P}) \to \mathsf{Mod}(\mathsf{Q})$ *preserves filtered colimits.*

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